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INVARIANT POLYNOMIALS ON LIE ALGEBRAS OF INHOMOGENEOUS UNITARY AND SPECIAL ORTHOGONAL GROUPS

BY

S. J. TAKIFF

ABSTRACT. The ring of invariant polynomials for the adjoint action of a Lie group on its Lie algebra is described for the inhomogeneous unitary and special orthogonal groups. In particular a new proof is given for the fact that this ring for the inhomogeneous Lorentz group is generated by two algebraically independent homogeneous polynomials of degrees two and four.

During a series of lectures given at the Indian Statistical Institute in 1965, V. S. Varadarajan proved the following result:

> The ring of invariant polynomials for the adjoint action of the inhomogeneous special Lorentz group is generated by two algebraically independent homogeneous polynomials of degrees two and four.

In this paper a technique used in the above is employed to obtain another proof of this result. More generally the above ring for the inhomogeneous unitary and special orthogonal groups is essentially described. A method due to J. Rosen is used to obtain invariant polynomials for the inhomogeneous groups from those of the homogeneous groups of next higher dimension.

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1. We now present some definitions, notation and preliminary results. Let $\pi: G \longrightarrow GL(V)$ be a representation of a group G on a finite dimensional vector space V. Let $S(V)$ denote the symmetric algebra of V and recall that $S(V) = \bigoplus_{n=0}^{\infty} S_n(V)$, where $S_n(V)$ is the subspace of homogeneous polynomials of degree n. We shall let $I(G, V, \pi)$ denote the algebra of polynomials in $S(V)$ invariant under the induced action of G on $S(V)$; note that $I(G, V, \pi) = \bigoplus_{n=0}^{\infty}\{S_n(V) \cap I(G, V, \pi)\}$. Now let V^* be the dual space of V and let $P(V) =$

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$S(V^*)$, the algebra of polynomial functions on V. We define an action of G on $P(V)$ as follows:

$$\pi(g)p(v) = p(\pi(g^{-1})v), \qquad p \in P(V),\ v \in V,\ g \in G.$$

Then denote the algebra of all $p \in P(V)$ invariant under this action by $I^*(G, V, \pi)$.

Now let B be a nondegenerate bilinear form on V. We can identify V^* with V via B by letting $v \longrightarrow v^*$ where $v^*(w) = B(w, v)$, $v, w \in V$. Moreover, we define $\pi^*: G \longrightarrow GL(V^*)$, the representation contragradient to π, by letting

$$\pi^*(g)v^*(w) = v^*(\pi(g^{-1})w) = B(\pi(g^{-1})w, v), \qquad g \in G, \quad v, w \in V.$$

Finally, let V^{**} be the dual space of V^* and identify V^{**} with V via B by letting $v \longrightarrow v^{**}$ where $v^{**}(w^*) = w^*(v) = B(v, w)$, $v, w \in V$. Then the following is immediate.

Proposition 1.1. *Let $\{v_1, \cdots, v_n\}$ be a basis for V. Let ϕ: $S(V) \longrightarrow P(V^*)$ be defined by letting $\phi(p(v_1, \cdots, v_n)) = p(v_1^{**}, \cdots, v_n^{**})$. Then ϕ is an isomorphism and it restricts to an isomorphism of $I(G, V, \pi)$ onto $I^*(G, V^*, \pi^*)$.*

Now let G be a connected Lie group with Lie algebra L and let Ad be the adjoint representation of G on L. Then we shall let $I(L) = I(G, L, \mathrm{Ad})$ and $I^*(L) = I^*(G, L^*, \mathrm{Ad}^*)$. Also for $x, y \in L$ let $\mathrm{ad}(x)y = [x, y]$. Then ad extends uniquely to a derivation ad: $S(L) \longrightarrow \mathrm{Der}(S(L))$, the algebra of derivations of $S(L)$ with $\mathrm{ad}(w)z = -\mathrm{ad}(z)w$ for $w, z \in S(L)$. It is easy to see that $I(L) = \{w \in S(L) | \mathrm{ad}(x)w = 0 \text{ for all } x \in L\} = \{w \in S(L) | \mathrm{ad}(z)w = 0 \text{ for all } z \in S(L)\}$.

Proposition 1.2. *Let $L = A \oplus T$ be a Levi decomposition of L where the radical A of L is abelian. Then*

$$I(L) = \bigoplus_{i, j=0}^{\infty} \{(S_i(A) \otimes S_j(T)) \cap I(L)\}.$$

Proof. Let $p \in S_n(L) \cap I(L)$; then $p = \sum_{j=0}^{n} p_j$ where $p_j \in S_j(A) \otimes S_{n-j}(T)$. Now for any $x \in L$, we have $0 = \mathrm{ad}(x)p = \sum_{j=0}^{n} \mathrm{ad}(x)p_j$. If additionally $x \in A$ then $\mathrm{ad}(x)p_n = 0$ and $\mathrm{ad}(x)p_j \in S_{j+1}(A) \otimes S_{n-j-1}(T)$, $0 \leq j \leq n-1$. While if $x \in T$ then $\mathrm{ad}(x)p_j \in S_j(A) \otimes S_{n-j}(T)$, $0 \leq j \leq n$. It is then clear that the p_j are also in $I(L)$.

The following construction is used repeatedly. Let η: $G \longrightarrow GL(V)$ be an analytic representation of a Lie group on a finite dimensional vector space. We view V as an abelian Lie group and form the semidirect product $V \times_\eta G$ where $(v, g) \circ (v', g') = (v + \eta(g)v', gg')$, $v, v' \in V$, $g, g' \in G$. Let L be the Lie algebra of G with Lie product $[\ ,\ ]_L$ and canonically identify the Lie algebra of V with itself. Then the Lie algebra of $V \times_\eta G$ is the vector space direct sum $V \oplus L$ with Lie product $[v, v'] = 0$, $[x, v] = d\eta(x)v$ and $[x, x'] = [x, x']_L$, where

$v, v' \in V$, $x, x' \in L$ and $d\eta$ is the differential of η. We will denote this Lie algebra by $V \oplus_\eta L$.

If $G \subset GL(n, F)$ ($F = \mathbf{R}$ or $\mathbf{C}$) then we shall always let $\rho\colon G \to GL(F^n)$ be as follows:

$$\rho(A)\,{}^t(x_1, \cdots, x_n) = A\,{}^t(x_1, \cdots, x_n), \qquad A \in G,\ {}^t(x_1, \cdots, x_n) \in F^n.$$

2. Recall $SO(n, \mathbf{C}) = \{A \in GL(n, \mathbf{C}) \mid {}^tAA = 1$ and $\det A = 1\}$; its Lie algebra $\mathfrak{so}(n, \mathbf{C}) = \{M \in \mathfrak{gl}(n, \mathbf{C}) \mid M = -{}^tM\}$ [3, p. 341]. Assume that $n \geq 3$ and let $\mathbf{C}^n$ denote the complex n-dimensional vector space with basis $\{e_1, \cdots, e_n\}$ where e_j is the column vector with 1 in the jth row and zeros elsewhere. Now form $\mathbf{C}^n \times_\rho SO(n, \mathbf{C})$ with Lie algebra $\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C})$. Let M_{ab} be the $n \times n$ matrix with -1 in the (a, b) position, 1 in the (b, a) position and zeros elsewhere. Then $M_{ab} = -M_{ba}$ and $\{e_1, \cdots, e_n\} \cup \{M_{ab} \mid 1 \leq a < b \leq n\}$ is a basis for $\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C})$; moreover, for $1 \leq a, b, c, d \leq n$,

$$[M_{ab}, M_{cd}] = \delta_{ac}M_{bd} - \delta_{bc}M_{ad} + \delta_{ad}M_{cb} - \delta_{bd}M_{ca},$$

$$[M_{ab}, e_c] = \delta_{ac}e_b - \delta_{bc}e_a \quad \text{and} \quad [e_a, e_b] = 0,$$

where δ is the Kronecker delta function. The following is clear.

Proposition 2.1. $p_0 = \sum_{a=1}^n e_a^2 \in I(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$.

We now pursue Rosen's technique [5] for obtaining invariants in $I(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$ from those in $I(\mathfrak{so}(n+1, \mathbf{C}))$. Let

$$T = S(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))\,[(-p_0)^{-1/2}];$$

here $(-p_0)^{-1/2}$ and thus T are imbedded in the algebraic closure of the quotient field of $S(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$. In T define

$$\widetilde{M}_{ab} = M_{ab}, \qquad 1 \leq a, b \leq n,$$

$$\widetilde{M}_{a,n+1} = -\widetilde{M}_{n+1,a} = (-p_0)^{-1/2}J_a \quad \text{where } J_a = \sum_{b=1}^{n} e_b M_{ab},\ 1 \leq a \leq n-1,$$

$$\widetilde{M}_{n+1,n+1} = 0.$$

Now by Proposition 2.1 ad can be uniquely extended to a derivation $\operatorname{ad}\colon T \to \operatorname{Der}(T)$; and a lengthy computation shows the following to be true.

Proposition 2.2.

$$\operatorname{ad}(\widetilde{M}_{ab})\widetilde{M}_{cd} = \delta_{ac}\widetilde{M}_{bd} - \delta_{bc}\widetilde{M}_{ad} + \delta_{ad}\widetilde{M}_{cb} - \delta_{bd}\widetilde{M}_{ca}, \qquad 1 \leq a, b, c, d \leq n+1.$$

Proposition 2.3. *Let* $q \in S(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$. *Assume that* $\operatorname{ad}(M_{ab})q = 0$, $1 \leq a < b \leq n$, *and* $\operatorname{ad}(J_a)q = 0$, $1 \leq a \leq n-1$. *Then* $q \in I(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$.

Proof. We need to show that $\operatorname{ad}(e_b)q = 0$, $1 \leq b \leq n$. So we compute for

$1 \le a \le n$, $0 = \mathrm{ad}(J_a)q = \sum_{b=1}^n \mathrm{ad}(e_b M_{ab})q = \sum_{b=1}^n e_b(\mathrm{ad}(M_{ab})q) + M_{ab}(\mathrm{ad}(e_b)q) = \sum_{b=1}^n M_{ab}(\mathrm{ad}(e_b)q)$.

Moreover by Proposition 2.1, $0 = \mathrm{ad}(p_0)q = \sum_{b=1}^n 2e_b(\mathrm{ad}(e_b)q)$. However,

$$\det \begin{pmatrix} -M_{12} & 0 & M_{23} & \cdots & M_{2n} \\ -M_{13} & -M_{23} & 0 & \cdots & M_{3n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ -M_{1n} & -M_{2n} & -M_{3n} & \cdots & 0 \\ e_1 & e_2 & e_3 & \cdots & e_n \end{pmatrix}$$

$$= (-M_{12})(-M_{23})\cdots(-M_{n-1,n})(e_n) + \text{other terms} \neq 0.$$

Therefore $\mathrm{ad}(e_b)q = 0$, $1 \le b \le n$.

Now for $1 \le j \le [(n-1)/2]$ ($[(n-1)/2]$ being the greatest integer less than or equal to $(n-1)/2$) let

$$p_{n,2j} = \sum_{1 \le a_1, \cdots, a_{2j} \le n} M_{a_1 a_2} M_{a_2 a_3} \cdots M_{a_{2j-1} a_{2j}} M_{a_{2j} a_1}.$$

And if n is even let

$$p_n = \sum_\sigma \Delta(\sigma) M_{\sigma(1)\sigma(2)} M_{\sigma(3)\sigma(4)} \cdots M_{\sigma(n-1)\sigma(n)}$$

where the sum is taken over all permutations σ of the set $\{1, 2, \cdots, n\}$ with $\Delta(\sigma) = 1$ (-1) if the permutation σ is even (odd). Then it is well known (see [2], [4], or [5]) that $I(\mathfrak{so}(n, \mathbf{C}))$ equals $\mathbf{C}[p_{n,2}, \cdots, p_{n,n-1}]$ if n is odd and it equals $\mathbf{C}[p_n, p_{n,2}, \cdots, p_{n,n-2}]$ if n is even. Moreover, the polynomials $p_{n,2}, \cdots, p_{n,n-1}$ $(p_n, p_{n,2}, \cdots, p_{n,n-2})$ are algebraically independent over $\mathbf{C}$, for n odd (even).

Finally for $1 \le j \le [(n-2)/2]$ let

$$\hat{p}_{n,2j} = (-p_0)^j \sum_{1 \le a_1, \cdots, a_{2j} \le n+1} \widetilde{M}_{a_1 a_2} \widetilde{M}_{a_2 a_3} \cdots \widetilde{M}_{a_{2j-1} a_{2j}} \widetilde{M}_{a_{2j} a_1}.$$

Then by Propositions 2.2 and 2.3, the $\hat{p}_{n,2j}$ are in $I(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$.

3. Let $U(n) = \{A \in GL(n, \mathbf{C}) | A\, {}^t\bar{A} = 1\}$ with Lie algebra $\mathfrak{u}(n) = \{M \in \mathfrak{gl}(n, \mathbf{C}) | M = -{}^t\bar{M}\}$ [3, p. 341]. Again assume that $n \ge 3$ and form $\mathbf{C}^n \times_\rho U(n)$ with Lie algebra $\mathbf{C}^n \oplus_\rho \mathfrak{u}(n)$. Now let V be a complex vector space with $\dim_{\mathbf{C}} V = 2n$ and let $\{v_1, \cdots, v_{2n}\}$ be a basis for V. Let E_{ab} be the $n \times n$ matrix with 1 in the (a, b) position and zeros elsewhere; then $\{E_{ab} | 1 \le a, b \le n\}$ is a basis for $\mathfrak{gl}(n, \mathbf{C})$. We now form the vector space direct sum $V \oplus \mathfrak{gl}(n, \mathbf{C})$ and make it into a Lie algebra by letting

$$[E_{ab}, E_{cd}] = \delta_{bc}E_{ad} - \delta_{ad}E_{cb},$$
$$[E_{ab}, v_c] = \tfrac{1}{2}\delta_{ac}(-v_b - \sqrt{-1}\,v_{n+b}) + \tfrac{1}{2}\delta_{bc}(v_a - \sqrt{-1}\,v_{n+a}),$$
$$[E_{ab}, v_{n+c}] = \tfrac{1}{2}\delta_{ac}(\sqrt{-1}\,v_b - v_{n+b}) + \tfrac{1}{2}\delta_{bc}(\sqrt{-1}\,v_a + v_{n+a}),$$
$$[v_a, v_b] = 0, \quad 1 \le a, b, c, d \le n.$$

It is easily checked that this Lie algebra is isomorphic to the complexification of the real Lie algebra $\mathbf{C}^n \oplus_\rho \mathfrak{u}(n)$. Moreover if π is the analytic homomorphism of $GL(n, \mathbf{C})$ into $GL(V)$ such that $d\pi(M)(v) = [M, v]$ for $M \in \mathfrak{gl}(n, \mathbf{C})$ and $v \in V$, then it is clear that $V \times_\pi GL(n, \mathbf{C})$ is a connected Lie group with Lie algebra $V \oplus_\pi \mathfrak{gl}(n, \mathbf{C})$ equal to the Lie algebra just constructed. The following is immediate.

Proposition 3.1. $q_0 = \sum_{a=1}^{2n} v_a^2 \in I(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$.

Now let $T' = S(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))\,[(-q_0)^{-1/2}]$. In T' we define, for $1 \le a, b \le n$,

$$\widetilde{E}_{ab} = E_{ab}, \quad \widetilde{E}_{a,n+1} = (-q_0)^{-1/2}K_a, \quad \widetilde{E}_{n+1,a} = (-q_0)^{-1/2}K_{n+a},$$
$$\widetilde{E}_{n+1,n+1} = (-q_0)^{-1}\sum_{a=1}^{n}(\sqrt{-1}\,v_a + v_{n+a})K_{n+a},$$

where

$$K_a = \sum_{b=1}^{n}(\sqrt{-1}\,v_b + v_{n+b})E_{ab} \quad \text{and} \quad K_{n+a} = \sum_{b=1}^{n}(-\sqrt{-1}\,v_b + v_{n+b})E_{ba}.$$

We see from Proposition 3.1 that ad can be uniquely extended to a derivation ad: $T' \to \mathrm{Der}\,(T')$; and a lengthy computation establishes the following.

Proposition 3.2. $\mathrm{ad}(\widetilde{E}_{ab})\,\widetilde{E}_{cd} = \delta_{bc}\,\widetilde{E}_{ad} - \delta_{ad}\,\widetilde{E}_{cb}$, $1 \le a, b, c, d \le n+1$.

Proposition 3.3. *Let* $q \in S(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$. *Assume that* $\mathrm{ad}(E_{ab})q = 0$, $1 \le a, b \le n$, *and* $\mathrm{ad}(K_a)q = 0$, $1 \le a \le 2n$. *Then* $q \in I(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$.

Proof. We need to show that $\mathrm{ad}(v_b)q = 0$, $1 \le b \le 2n$. Now for $1 \le a \le n$ we compute

$$0 = \mathrm{ad}\,(K_a)q = \mathrm{ad}\left(\sum_{b=1}^{n}(\sqrt{-1}\,v_b + v_{n+b})E_{ab}\right)q$$
$$= \sum_{b=1}^{n} E_{ab}(\mathrm{ad}\,(\sqrt{-1}\,v_b + v_{n+b})q) + (\sqrt{-1}\,v_b + v_{n+b})(\mathrm{ad}\,(E_{ab})q)$$
$$= \sum_{b=1}^{n} E_{ab}(\sqrt{-1}\,\mathrm{ad}\,(v_b)q + \mathrm{ad}\,(v_{n+b})q).$$

Similarly, for $1 \leq a \leq n$,

$$0 = \mathrm{ad}(K_{n+a})q = \sum_{b=1}^{n} E_{ab}(-\sqrt{-1}\,\mathrm{ad}(v_b)q + \mathrm{ad}(v_{n+b})q).$$

However,

$$\det\begin{pmatrix} E_{11} & \cdots & E_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ E_{n1} & \cdots & E_{nn} \end{pmatrix} = E_{11}E_{22}\cdots E_{nn} + \text{other terms} \neq 0;$$

and thus $\mathrm{ad}(v_b)q = 0 = \mathrm{ad}(v_{n+b})q$, $1 \leq b \leq n$.

Now let D be the matrix

$$\begin{pmatrix} E_{11} & \cdots & E_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ E_{n1} & \cdots & E_{nn} \end{pmatrix}$$

and let

$$q_{n,j} = \mathrm{tr}(D^j) = \sum_{1 \leq a_1, \cdots, a_j \leq n} E_{a_1a_2}E_{a_2a_3}\cdots E_{a_{j-1}a_j}E_{a_ja_1}, \qquad 1 \leq j \leq n,$$

where tr( ) denotes the trace of ( ). Then it follows from [2] or [4] that $I(\mathfrak{gl}(n, \mathbf{C})) = \mathbf{C}[q_{n,1}, q_{n,2}, \cdots, q_{n,n}]$. Finally let

$$\hat{q}_{n,j} = (q_0)^j \sum_{1 \leq a_1, \cdots, a_j \leq n+1} \tilde{E}_{a_1a_2}\tilde{E}_{a_2a_3}\cdots\tilde{E}_{a_{j-1}a_j}\tilde{E}_{a_ja_1},$$

$1 \leq j \leq n-1$. Then by Propositions 3.2 and 3.3 we see that the $\hat{q}_{n,j}$ are in $I(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$.

4. In this section we present the result from Varadarajan's lectures that we need. Let (F^n, G, L) denote either $(\mathbf{R}^n, SO(n, \mathbf{R}), \mathfrak{so}(n, \mathbf{R}))$ or $(\mathbf{C}^n, U(n), \mathfrak{u}(n))$, with $n \geq 3$. Form $F^n \times_\rho G$ with Lie algebra $F^n \oplus_\rho L$. Now for $x = \sum_{i=1}^n x_ie_i$, $y = \sum_{i=1}^n y_ie_i \in F^n$ let $\langle x, y\rangle = \sum_{i=1}^n (x_i\bar{y}_i + \bar{x}_iy_i)$, and note that $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $A \in G$. Finally let B be the following bilinear form on $F^n \oplus L$:

$$B((x, M), (y, N)) = \langle x, y\rangle + \mathrm{tr}({}^tMN), \qquad x, y \in F^n, \quad M, N \in L.$$

It is clear that B is nondegenerate and we can then identify $(F^n \oplus_\rho L)^*$ with $F^n \oplus_\rho L$ via B.

Now if $x, y \in F^n$ then $M \longrightarrow \langle Mx, y\rangle$ is a real linear transformation of L into R. Furthermore, if $\phi(N)(M) = \mathrm{tr}({}^tMN)$, $M, N \in L$, then ϕ is a bijective real linear transformation from L onto L^*. Consequently, there exists a unique $\lambda(x, y) \in L$ such that $\langle Mx, y\rangle = \mathrm{tr}({}^tM\lambda(x, y))$ for all $M \in L$.

Proposition 4.1. *Let* $x = \sum_{i=1}^n x_i e_i$, $y = \sum_{i=1}^n y_i e_i \in F^n$. *Then* $\lambda(x, y)$ *is the matrix with* $x_j\bar{y}_i - \bar{x}_i y_j$ *in the* (i, j) *position.*

Proof. Let $M = (m_{ij})$ be an arbitrary matrix in L and let $\lambda(x, y) = (c_{ij})$. Since $\mathrm{tr}({}^tM\lambda(x, y)) = \sum_{i,j=1}^n m_{ij}c_{ij}$ and $\langle Mx, y\rangle = \sum_{i,j=1}^n m_{ij}x_j\bar{y}_i$, we see that $c_{ij} = x_j\bar{y}_i - \bar{x}_i y_j$, $1 \le i, j \le n$.

Proposition 4.2. *Let* $x, y \in F^n$, $A \in G$ *and* $M \in L$. *Then* $\mathrm{Ad}(x, A)(y, M) = (Ax - AMA^{-1}x, AMA^{-1})$ *and* $\mathrm{Ad}^*(x, A)(y, M)^* = (Ay, \lambda(x, Ay) + {}^tA^{-1}M^tA)^*$.

Proof. The first formula follows immediately, since

$$\mathrm{Ad}(x, A)(y, M) = (d/dt)\{(x, A)\circ \exp t(y, M) \circ (x, A)^{-1}\}_{t=0}$$
$$= (Ax - AMA^{-1}x, AMA^{-1}).$$

To show the second, let $z \in F^n$ and $N \in L$. We compute

$$(\mathrm{Ad}^*(x, A)(y, M)^*)(z, N) = B(\mathrm{Ad}(x, A)^{-1}(z, N), (y, M))$$
$$= B((A^{-1}z + A^{-1}Nx, A^{-1}NA), (y, M)) = \langle A^{-1}z + A^{-1}Nx, y\rangle + \mathrm{tr}({}^t(A^{-1}NA)M)$$
$$= \langle z + Nx, Ay\rangle + \mathrm{tr}({}^tA^tN^tA^{-1}M) = \langle z, Ay\rangle + \langle Nx, Ay\rangle + \mathrm{tr}({}^tA^tN^tA^{-1}M)$$
$$= \langle z, Ay\rangle + \mathrm{tr}({}^tN\lambda(x, Ay)) + \mathrm{tr}({}^tN^tA^{-1}M^tA)$$
$$= B((z, N), (Ay, \lambda(x, Ay) + {}^tA^{-1}M^tA)) = (Ay, \lambda(x, Ay) + {}^tA^{-1}M^tA)^*(z, N).$$

Note. In the sequel we shall omit "*" and "**" from elements in dual spaces.

Let $W = \{(y, M) \in (F^n \oplus_\rho L)^* | y \neq 0\}$ and let $U = \{(y, M) \in (F^n \oplus_\rho L)^* | y = y_n e_n,\ 0 < y_n \in R$ and $M = (m_{ij})$ with $m_{nj} = 0 = m_{jn},\ 1 \le j \le n\}$.

Proposition 4.3. $W \subset \{\mathrm{Ad}^*(x, A)U | (x, A) \in F^n \times_\rho G\}$.

Proof. Let $(y, M) \in W$ and let $\alpha = (\langle y, y\rangle/2)^{1/2} > 0$. Then $\alpha^{-1}y$ is a vector in F^n of unit length (with respect to the usual Euclidean metric on F^n). Hence there exists $A \in G$ such that $Ae_n = \alpha^{-1}y$ [1, p. 11], and so $A^{-1}y = \alpha e_n$. Now let ${}^tAM^tA^{-1} = (m_{ij}) \in L$ and choose $x = \sum_{j=1}^n x_j e_j \in F^n$ such that $x_j = -m_{nj}/\alpha$, $1 \le j \le n-1$, and $x_n = -m_{nn}/2\alpha$. Then $\mathrm{Ad}^*(x, A^{-1})(y, M) = (A^{-1}y, \lambda(x, A^{-1}y) + {}^tAM^tA^{-1}) = (\alpha e_n, \lambda(x, \alpha e_n) + (m_{ij})) \in U$. Consequently, $(y, M) \in \mathrm{Ad}^*(x, A^{-1})^{-1}U$ and so $W \subset \{\mathrm{Ad}^*(x, A)U | (x, A) \in F^n \times_\rho G\}$.

5. We now apply the results of §4. So let $x = \sum_{j=1}^n x_j e_j$, $y = \sum_{j=1}^n y_j e_j \in \mathbf{C}^n$, let $M, N \in \mathfrak{so}(n, \mathbf{C})$ and let

$$B'((x, M), (y, N)) = \sum_{j=1}^n (x_j\bar{y}_j + \bar{x}_j y_j) + \mathrm{tr}({}^tMN).$$

Further, for $x = \sum_{j=1}^{2n} x_j v_j$, $y = \sum_{j=1}^{2n} y_j v_j \in V$ and $M, N \in \mathfrak{gl}(n, \mathbf{C})$ let

$$B''((x, M), (y, N)) = 2\left(\sum_{j=1}^{n} (x_j y_j + x_{n+j} y_{n+j})\right) + \mathrm{tr}({}^t MN).$$

Then B' (B'') is a nondegenerate bilinear form on $\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C})$ ($V \oplus_\pi \mathfrak{gl}(n, \mathbf{C})$) and we can identify $(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))^*$ ($(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))^*$) with $\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C})$ ($V \oplus_\pi \mathfrak{gl}(n, \mathbf{C})$) via B' (B'').

Now let $W' = \{(y, M) \in (\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))^* \mid 0 \neq y \in \mathbf{R}^n, M \in \mathfrak{so}(n, \mathbf{R})\}$ and let $U' = \{(y, M) \in (\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))^* \mid y = y_n e_n, 0 < y_n \in \mathbf{R}, M = (m_{ij})$ with $m_{nj} = 0 = m_{jn}, 1 \leq j \leq n\}$. Moreover, let $W'' = \{(y, M) \in (V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))^* \mid 0 \neq y \in \sum_{j=1}^{2n} \mathbf{R} v_j$, $M \in \mathfrak{u}(n)\}$ and let $U'' = \{(y, M) \in (V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))^* \mid y = y_n e_n + y_{2n} e_{2n}, 0 < y_n = y_{2n} \in \mathbf{R}, M = (m_{ij})$ with $m_{nj} = 0 = m_{jn}, 1 \leq j \leq n\}$. Then Proposition 4.3 yields the following result.

Proposition 5.1. $W' \subset \{\mathrm{Ad}^*(x, A)U' \mid (x, A) \in \mathbf{C}^n \times_\rho SO(n, \mathbf{C})\}$ *and* $W'' \subset \{\mathrm{Ad}^*(x, A)\, U'' \mid (x, A) \in V \times_\pi GL(n, \mathbf{C})\}$.

Corollary 5.2. *Let* $p, q \in I^*(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$ ($I^*(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$) *and assume that* $p|U' = q|U'$ ($p|U'' = q|U''$). *Then* $p = q$.

Proof. By Proposition 5.1, $p - q|W' = 0$ ($p - q|W'' = 0$). Since W' (W'') is C-Zariski dense in $(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))^*$ ($(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))^*$), we see that $p = q$.

Proposition 5.3. *Let* $p \in I^*(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$ *and let* $q \in I^*(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$.

1. $p|U' \in \mathbf{C}[e_n, p_{n-1}, p_{n-1,2}, p_{n-1,4}, \cdots, p_{n-1,n-3}]$ *if* n *is odd.*
2. $p|U' \in \mathbf{C}[e_n, p_{n-1,2}, p_{n-1,4}, \cdots, p_{n-1,n-2}]$ *if* n *is even.*
3. $q|U'' \in \mathbf{C}[v_n, q_{n-1,1}, q_{n-1,2}, \cdots, q_{n-1,n-2}]$.

Here we identify in the natural way $I^*(\mathfrak{so}(n-1, \mathbf{C}))$ and $I^*(\mathfrak{gl}(n-1, \mathbf{C}))$ as subalgebras of $P((\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))^*)$ and $P((V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))^*)$ respectively.

Proof. By Proposition 1.2 we can assume $p \in (P_a(\mathbf{C}^{n*}) \otimes P_b(\mathfrak{so}(n, \mathbf{C})^*)) \cap I^*(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$ and $q \in (P_c(\mathbf{C}^{n*}) \otimes P_d(\mathfrak{gl}(n, \mathbf{C})^*)) \cap I^*(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$. Then $p|U' = (e_n)^a p'$ and $q|U'' = (v_n)^c q'$ where $p' \in I^*(\mathfrak{so}(n-1, \mathbf{C}))$ and $q' \in I^*(\mathfrak{gl}(n-1, \mathbf{C}))$. The proposition then follows from the statements at the end of §§2 and 3.

Corollary 5.4. *Let* n *be odd. Then* $p_0, \hat{p}_{n,2}, \hat{p}_{n,4}, \cdots, \hat{p}_{n,n-3}$ *are in* $I^*(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$ *and are algebraically independent over* $\mathbf{C}$. *Therefore*

$$(n-1)/2 \leq \mathrm{tran.deg.}_{\mathbf{C}}(I^*(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))) \leq (n+1)/2.$$

Proof. If $Q(p_0, \hat{p}_{n,2}, \cdots, \hat{p}_{n,n-3}) = 0$ for $Q \in \mathbf{C}[X_1, \cdots, X_{(n-1)/2}]$ then

$$0 = Q(p_0, \hat{p}_{n,2}, \cdots, \hat{p}_{n,n-3})|U' = Q(e_n^2, p_{n-1,2}, \cdots, p_{n-1,n-3}).$$

Since $e_n, p_{n-1,2}, \cdots, p_{n-1,n-3}$ are algebraically independent over C then $Q = 0$ and we are done.

Recall that $n \geq 3$ and let $m \geq 2$.

Proposition 5.5. *Let* $0 \neq p \in (P_a(\mathbf{C}^{2m*}) \otimes P_b(\mathfrak{so}(2m, \mathbf{C})^*)) \cap I^*(\mathbf{C}^{2m} \oplus_\rho \mathfrak{so}(2m, \mathbf{C}))$ *and let* $0 \neq q \in (P_c(V^*) \otimes P_d(\mathfrak{gl}(n, \mathbf{C})^*)) \cap I^*(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$. *Then a, b and c are even.*

Proof. Choose a positive integer t such that $a + 2t \geq b$. If a is odd then

$$((p_0)^{1/2}(p_0)^t p)|U' = Q(e_{2m}^2, p_{2m-1,2}, \cdots, p_{2m-1,2m-2})$$
$$= Q(p_0, \hat{p}_{2m,2}, \cdots, \hat{p}_{2m,2m-2})|U', \qquad Q \in \mathbf{C}[x_1, \cdots, X_m].$$

Consequently, $((p_0)^{2t+1}p^2)|U' = Q(p_0, \hat{p}_{2m,2}, \cdots, \hat{p}_{2m,2m-2})^2|U'$ and so by Corollary 5.2, $(p_0)^{2t+1}p^2 = Q(p_0, \hat{p}_{2m,2}, \cdots, \hat{p}_{2m,2m-2})^2$. But $P((\mathbf{C}^n \oplus_\rho \mathfrak{so}(2m, \mathbf{C}))^*)$ is a unique factorization domain and p_0 is irreducible. Thus since p_0 appears an odd (even) number of times in a factorization of $(p_0)^{2t+1}p^2$ $(Q(p_0, \hat{p}_{2m,2}, \cdots, \hat{p}_{2m,2m-2})^2)$, we have a contradiction. Hence a is even and it is clear that b is also even. A similar argument shows that c is even.

The following theorem should now be clear.

Theorem 5.6. *Let* $p \in (P_{2a}(\mathbf{C}^{2m*}) \otimes P_{2b}(\mathfrak{so}(2m, \mathbf{C})^*)) \cap I^*(\mathbf{C}^{2m} \oplus_\rho \mathfrak{so}(2m, \mathbf{C}))$ *and let* $q \in (P_{2c}(V^*) \otimes P_d(\mathfrak{gl}(n, \mathbf{C})^*)) \cap I^*(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$. *Let* $t =$ maximum $\{0, b - a\}$ *and let* $s =$ maximum $\{0, d - c\}$. *Then* $(p_0)^t p \in \mathbf{C}[p_0, \hat{p}_{2m,2}, \hat{p}_{2m,4}, \cdots, \hat{p}_{2m,2m-2}]$ *and* $(q_0)^s q \in \mathbf{C}[q_0, \hat{q}_{n,1}, \hat{q}_{n,2}, \cdots, \hat{q}_{n,n-1}]$.

Remark. We note that as in the proof of Corollary 5.4, $p_0, \hat{p}_{2m,2}, \cdots, \hat{p}_{2m,2m-2}$ are algebraically independent over C and so are $q_0, \hat{q}_{n,1}, \cdots, \hat{q}_{n,n-1}$.

Corollary 5.7. $I^*(\mathbf{C}^4 \oplus_\rho \mathfrak{so}(4, \mathbf{C})) = \mathbf{C}[p_0, \hat{p}_{4,2}]$.

Proof. Let $p \in I^*(\mathbf{C}^4 \oplus_\rho \mathfrak{so}(4, \mathbf{C})) \cap P_{2a}((\mathbf{C}^4 \oplus_\rho \mathfrak{so}(4, \mathbf{C}))^*)$, and choose t minimal such that $(p_0)^t p \in \mathbf{C}[p_0, \hat{p}_{4,2}]$. Since p is homogeneous then $(p_0)^t p = p_0 Q(p_0, \hat{p}_{4,2}) + \alpha(\hat{p}_{4,2})^b$ for $Q \in \mathbf{C}[X_1, X_2]$, $\alpha \in \mathbf{C}$, and some positive integer b. Now if $t \neq 0$ then $\alpha \neq 0$; but then p_0 divides $\hat{p}_{4,2}$ in $P((\mathbf{C}^4 \oplus_\rho \mathfrak{so}(4, \mathbf{C}))^*)$. As this is not the case we must have $t = 0$ and $p \in \mathbf{C}[p_0, \hat{p}_{4,2}]$.

Remark. Unfortunately we cannot obtain similar corollaries for $I^*(\mathbf{C}^{2m} \oplus_\rho \mathfrak{so}(2m, \mathbf{C}))$, $m \geq 3$, and $I^*(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$, $n \geq 3$.

6. Now by (1.1) statements similar to those of (5.4), (5.6) and (5.7) can be be made for $I(\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C}))$ and $I(V \oplus_\pi \mathfrak{gl}(n, \mathbf{C}))$, $n \geq 3$. Moreover, since $\mathbf{C}^n \oplus_\rho \mathfrak{u}(n)$ is a real form of $V \oplus_\pi \mathfrak{gl}(n, \mathbf{C})$ the same can be said of $I(\mathbf{C}^n \oplus_\rho \mathfrak{u}(n))$. In fact in (5.4), (5.6) and (5.7) we can clearly replace $\mathbf{C}^n \times_\rho SO(n, \mathbf{C})$, $\mathbf{C}^n \oplus_\rho \mathfrak{so}(n, \mathbf{C})$, $V \times_\rho GL(n, \mathbf{C})$ and $V \oplus_\pi \mathfrak{gl}(n, \mathbf{C})$ by $\mathbf{R}^{a+b} \times_\rho SO(a, b)$, $\mathbf{R}^{a+b}$

$\oplus_\rho \mathfrak{so}(a, b)$, $\mathbf{C}^{a+b} \times_\rho U(a, b)$, $\mathbf{C}^{a+b} \oplus_\rho \mathfrak{u}(a, b)$, $a + b \geq 3$, since the latter are real forms of the former [3, pp. 339–355]. In particular, we see that Corollary 5.7 is true (with p_0 and $\hat{p}_{4,2}$ modified) for the inhomogeneous Lorentz group $\mathbf{R}^4 \times_\rho SO(1, 3)$ with Lie algebra $\mathbf{R}^4 \oplus_\rho \mathfrak{so}(1, 3)$.

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823